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Equivalence of PLL Systems and a Discriminator Followed by a Non-Linear Feedback Filter

by

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Abstract

A system consisting of an ideal bandpass limiter-discriminator followed by a non-linear feedback filter has been analyzed and found to be equivalent to a bandpass limiter followed by a phase-lock loop for a class of phase detector non-linearities. This is done for the cases of sine, tanlock and sawtooth nonlinearities.

Glossary

BPL	Bandpass Limiter
IF	Intermediate Frequency Amplifier
PLL	Phase Locked Loop
DISC	Discriminator
VCO	Voltage Controlled Oscillator

Introduction

Consider the systems shown in Fig. 1 and Fig. 2. Let each system have the same IF output signal.

$$e_{IF} = n(t) + \alpha \sin [\omega_o t + m(t)] \tag{1}$$

where $n(t)$ is a white band limited Gaussian noise, α is the carrier amplitude, ω_o is the carrier frequency and $\dot{m}(t)$ is the frequency modulation. If the modulation bandwidth is much less than the BPL bandwidth the noise can be expanded in terms of in phase and quadrature phase components with respect to the signal phase. That is

$$n(t) = n_1(t) \cos [\omega_o t + m(t)] + n_2(t) \sin [\omega_o t + m(t)] \tag{2}$$

Letting

$$A(t) = [n_1^2(t) + (\alpha + n_2(t)^2]^{\frac{1}{2}} \tag{3}$$

and

$$\gamma(t) = \tan^{-1} [n_1(t) \;/\; (n_2(t) + \alpha)] + m(t) \tag{4}$$

and substituting in (1), then

$$e_{IF} = A(t) \sin [\omega_o t + \gamma(t)] \tag{5}$$

McRae, Pelchat and Smith of Radiation Inc.[5] have proposed that the system shown in Fig. 1 is equivalent to the phase lock loop (PLL) when the system parameter g, the integrator gain, is unity. Fig. 2A is a PLL with preceding bandpass limiter (BPL). Fig. 2B is a PLL without BPL. They believe that the presence or lack of a BPL does not change the essential performance of a PLL.

However is Fig. 1 equivalent to Fig. 2A and Fig. 2B? More specifically, do the three systems have the same noise performance? It is easy to show that the three systems are equivalent for the high signal to noise ratio case, but it is not obvious whether they are for the general signal to noise ratio case.

The discussion that follows will answer these specific questions related to the above problem.

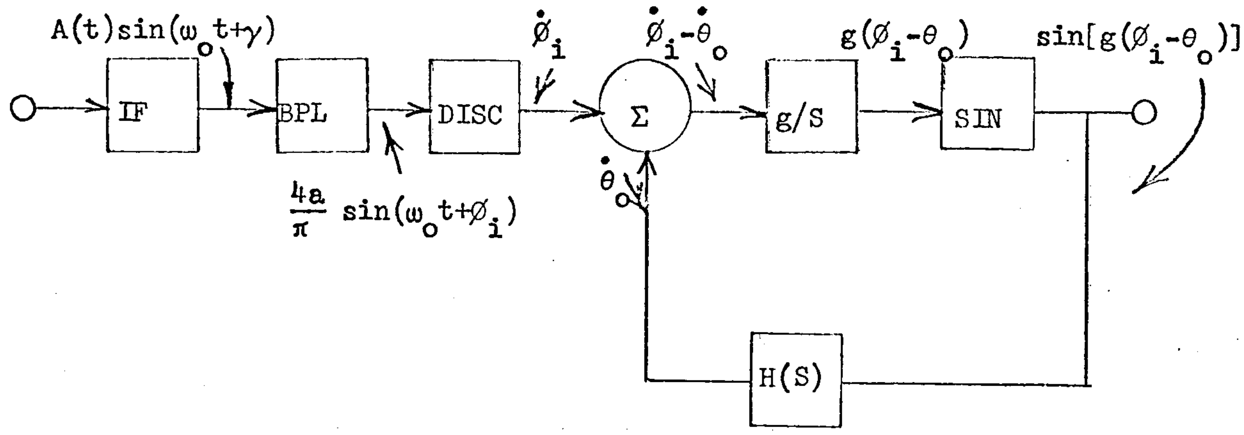

Fig. 1. McRae Et. Al. System.[5]

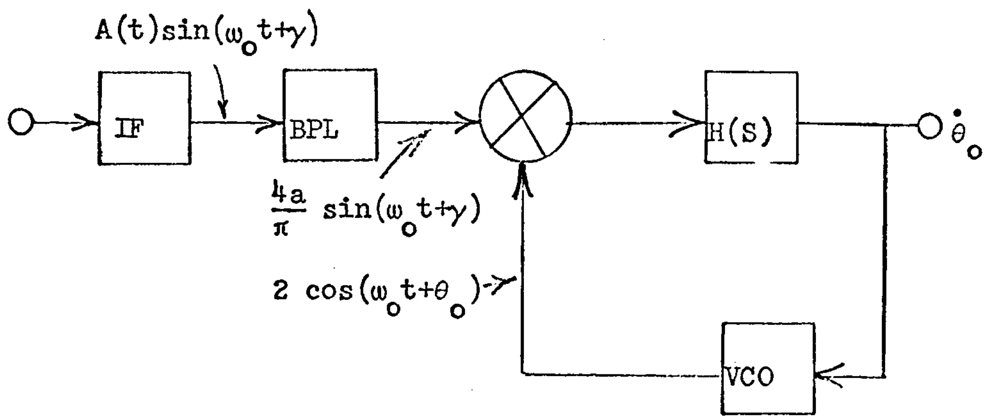

Fig. 2A. Standard PLL with BPL.

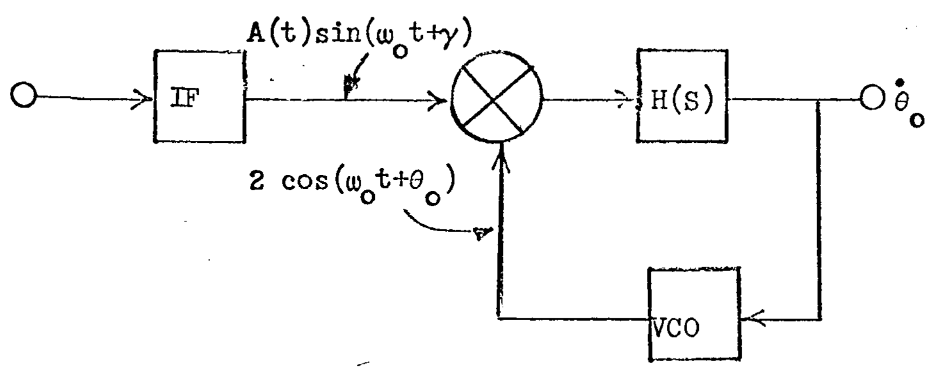

Fig. 2B. Standard PLL without BPL.

II. Analytical Techniques

Traditionally the noise performance of angle demodulators has been determined theoretically by obtaining an expression for the phase error variance. At first it was thought that this approach would be used to compare the noise performance of Figs. 1, 2A and 2B. The following analytical tools were considered:

1. Boonton's technique
2. Linear spectral approximation
3. Linear approximation
4. Voltera expansion
5. Pertibation expansion
6. Fokker Planck

Boonton's technique which was first applied to the analysis of PLL by Develet[2] requires that the noise at the input of the non-linear device be additive Gaussian. However the discriminator output noise is only Gaussian for high signal to noise ratios (S/N) and contains impulse noise at low S/N. Therefore the Gaussian requirement of the Boonton technique is violated in the region of interest.

The linear spectral approximation was used by Tausworthe.[10] This technique requires that the noise be additive. The noise at the output of the BPL is not additive since it is present only as phase jitter of the BPL output. (See pp. 8-9).

The linear approximation was rejected because under it McRae's system reduces to a discriminator followed by a linear filter which is known to have a higher threshold than the PLL. The linear model of the PLL shows no threshold, but the real PLL does have a threshold.

The Voltera expansion technique was used by Van Trees[12] to determine the threshold of a PLL. However, the technique is very involved and he states that it is even more difficult to apply to systems with non-Gaussian signals.

The pertibation expansion used by Margolis[4] requires that the noise be additive.

The Fokker Planck technique has been used by Viterbi[13]. It requires that the noise at the input be additive Gaussian. Furthermore, exact results are only tractable for the first order loop.

Consider Fig. 1. The noise at the discriminator output is in general not Gaussian. For low S/N it contains impulses and is only approximately Gaussian at moderate S/N.

Consider Fig. 2A. The noise at the BPL output is not Gaussian. Furthermore it is not additive (See p. (12)). The noise here appears as a phase jitter on a sine wave which has a constant envelope amplitude. (See pp.(8-9) for discussion of the BPL characteristics).

Since all of the above analysis techniques are either invalid or not tractable for calculating the phase-error variance, this approach was discarded in favor of a more direct approach.

Rather than compare the thresholds of systems obtained by analyzing their mathematical models, the models themselves will be analyzed and compared. It is reasoned that if the models are valid for all S/N and if the models are equivalent, then the systems will be equivalent for all S/N, and thus, they will have the same noise performance and threshold. That is, if the systems obey the same set of mathematical equations or equivalent sets of equations and have equivalent initial conditions, they will have the same responses to identical inputs. Therefore, they will have the same noise performance and threshold.

III. Model of McRae's System

The first step in this comparison is to obtain mathematical expressions describing McRae's system. This is done by obtaining or assuming expressions for each block in Fig. 1. Then this set of equations is manipulated to put it in the form of the equations obtained from Fig. 2A. The existence of this transformation is proof of equality of the systems.

The following assumptions are made concerning Fig. 1, 2A and 2B as well as for all subsequent diagrams.

1. The IF amplifiers in all diagrams have equal, ideal rectangular bandpasses with bandwidth B_{IF} and center frequency ω_o such that $B_{IF} << \omega_o$.

2. The BPL in all diagrams has an ideal zero hysteresis limiter which has an output

$$e_{\ell o} = \begin{cases} a & \text{for} \quad e_{\ell i} > 0 \\ 0 & \text{for} \quad e_{\ell i} = 0 \\ -a & \text{for} \quad e_{\ell i} < 0 \end{cases} \tag{6}$$

where $e_{\ell i}$ is the limiter input and a is the limiter amplitude constant. This is followed by an ideal bandpass filter with the same bandpass as the IF amplifier in 1 above.

3. The VCO in all diagrams is assumed to generate the output function,

$$e_{VCO} = 2 \cos \left\{ g \left[\omega_o t + \int_o^t \dot{\theta}_o (u)\, du + \theta_1 \right] \right\} \tag{7}$$

where $\dot{\theta}_o$ is the input, θ_1 is the initial phase and ω_o is the "center frequency."

4. The discriminator in all diagrams is assumed to be ideal. That is, each is assumed to provide the output function

$$e_{do} = \frac{d}{dt} \left\{ \sin^{-1} \left[\frac{\pi}{4a} e_{di}(t) \right] \right\} - \omega_o \tag{8}$$

in which it is assumed that $\sin^{-1}$ is multivalued. For

$$e_{di} = \frac{4a}{\pi} \sin(\omega_o t + \phi_i)$$

the discriminator output is $\dot{\phi}_i$. Note that because $\sin^{-1}$ is assumed to be multivalued, $\dot{\phi}_i$ is not calculated modulo 2π.

5. If $B_{IF} << \omega_o$ and the bandwidth of $H(S)$, the loop filter is $B_\ell << \omega_o$ and $h(t) = L^{-1} H(S)$ where L^{-1} denotes inverse Laplace transform, then

$$\left\{\sin[g(\phi_i - \theta_o)]\right\} * h(t) \cong \left\{2\cos[g(\omega_o t + \theta_o)] \cdot \sin[g(\omega_o t + \phi_i)]\right\} * h(t) \tag{10}$$

where * denotes convolution.

The proof of this follows:

Note that

$$2\cos[g(\omega_o t + \theta_o)] \cdot \sin[g(\omega_o t + \phi_i) = \sin[g(\phi_i - \theta_o)] + \sin[g(2\omega_o t + \phi_i + \theta_o)] \tag{11}$$

Since convolution is a linear operation, superposition applies and

$$(C + D) * h(t) = C * h(t) + D * h(t) \tag{12}$$

Since $H(S)$ is a low-pass filter and its bandwidth $B_\ell << \omega_o$ in most PLL then

$$|\sin[g(2\omega_o t + \phi_i + \theta_o)] * h| << |\sin[g(\phi_i - \theta_o)] * h| \tag{13}$$

and the former can be neglected. Q.E.D.

6. All of the assumptions 1-5 are valid for all S/N ratios.

IV. Noise Performance and Threshold Behaviour for g = 1

Consider McRae's system in Fig. 1 for the case of $g = 1$.[5] The non-linear filter following the discriminator is described by the function

$$\left\{\sin\left[\int_0^t (\dot{\phi}_i - \dot{\theta}_o)\,dt + \phi_1 - \theta_1\right]\right\} * h(t) = \dot{\theta}_o \tag{14}$$

with $\phi_1 = \phi_i(0)$ and $\theta_1 = \theta_o(0)$. This can be written

$$\left\{\sin(\phi_i - \theta_o)\right\} * h(t) = \dot{\theta}_o \tag{15}$$

Now if $B_{IF} << \omega_o$ and $B_\ell << \omega_o$ assumption 5 can be applied and

$$\left\{\sin(\omega_o t + \phi_i) \cdot \cos(\omega_o t + \theta_o)\right\} * h(t) \cong \dot{\theta}_o \tag{16}$$

for all S/N. Substituting

$$\theta_o = \int_o^t \dot{\theta}_o(\lambda)\, d\lambda + \theta_1 , \tag{17}$$

Then

$$2\left\{\sin(\omega_o t + \phi_i) \cdot \cos\left(\omega_o t + \int_o^t \dot{\theta}_o(\lambda)\, d\lambda + \theta_1\right)\right\} * h(t) \triangleq \dot{\theta}_o \tag{18}$$

Using assumption 3 for the case $g = 1$

$$\left\{\sin(\omega_o t + \phi_i) \cdot e_{VCO}\right\} * h(t) \cong \dot{\theta}_o \tag{19}$$

for all S/N.

Consider the IF amplifier and BPL in Fig. 1. It has been shown[14] that the output of the BPL in Fig. 1 under assumptions 1 and 2 is

$$e_{BPL} = 2a \cdot C(0, 1) \cdot \sin[\omega_o t + \gamma(t)] \tag{20}$$

where

$$C(0, 1) = [\Gamma(\tfrac{1}{2}) \cdot \Gamma(\tfrac{3}{2})]^{-1} = \frac{2}{\pi} \tag{21}$$

if the input is

$$e_{IF}' = A(t) \cdot \sin[\omega_o t + \gamma(t)]. \tag{22}$$

The BPL output for this input is

$$e_{BPL} = \frac{4a}{\pi} \cdot \sin[\omega_o t + \gamma(t)] \tag{23}$$

therefore

$$\phi_i(t) = \gamma(t) \tag{24}$$

One can ask if e_{BPL} can be expanded in terms of a pure signal component plus additive noise with some pdf. To determine this $\gamma(t)$ is split into signal and noise components

$$\gamma(t) = m(t) + \tan^{-1}[n_1(t) / (n_2(t) + \alpha)] \tag{25}$$

and from this e_{BPL} is expanded trigonometrically

$$e_{BPL} = \frac{4a}{\pi}\left\{\sin[\omega_o t + m(t)]\,\frac{n_2(t)+\alpha}{A(t)} + \cos[\omega_o t + m(t)]\,\frac{n_1(t)}{A(t)}\right\} \tag{26}$$

or

$$e_{BPL} = \frac{4a}{\pi\, A(t)} \left\{ n(t) + \alpha \sin\left[\omega_o t + m(t)\right] \right\} \tag{27}$$

Because $A^{-1}(t)$ is common to both of the terms in the curly brackets it's impossible to split e_{BPL} into separate signal and noise components. The conclusion is that the noise at the output of a BPL is not additive.

The discriminator by assumption 4 will operate on the BPL output to produce an output $\dot{\gamma}(t)$.

The result of these manipulations is the following set of equations.

$$\dot{\theta}_o = \left\{ e_{BPL}(t) \cdot e_{VCO}(t) \cdot \frac{\pi}{4a} \right\} * h(t) \tag{28}$$

where

$$e_{BPL} = \frac{4a}{\pi} \sin\left(\omega_o t + \gamma(t)\right) \tag{29}$$

and

$$e_{VCO} = 2 \cos\left[\omega_o t + \int_o^t \dot{\theta}_o(\lambda)\, d\lambda + \theta_1\right] \tag{30}$$

for the input

$$A(t) \sin\left[\omega_o t + \gamma(t)\right] \tag{31}$$

However, (28) is recognized as the equation describing the operation of a PLL with VCO described by (30) when supplied with an input signal e_{BPL}. Also (29) is recognized as the output of a BPL with input (31). Since this is exactly the system in Fig. 2A, and the application of the equations is valid for all S/N, it is concluded that Fig. 1 and Fig. 2A are equivalent and hence have the same noise performance and threshold behaviour under the stated assumptions.

V. Comparison of PLL With and Without BPL

The performances of Fig. 2A and Fig. 2B now are compared. The inclusion of the BPL in Fig. 2B will alter the signal level entering the loop from that existing in Fig. 2A, since the BPL produces a constant total power output. The signal level at the BPL output is a function of the S/N at its input.

Since the bandwidth B_ℓ and damping factor, ξ, of the loop depend on the input signal level and not the noise level, B_ℓ and ξ will vary with the BPL input S/N of Fig. 2A, and only with the signal in Fig. 2B. Therefore, in general, the threshold and noise performance of the two will be different.

A more meaningful comparison requires that a, the BPL gain, be controlled so that the multiplier input signal component of each PLL has the same amplitude. It may be possible under this condition to compare the loops by calculating and comparing their phase error variances. However, this would be a very difficult problem because of the non-Gaussian noise at the BPL output.

McRae[5] and others have claimed that experimental evidence shows there is no difference in the threshold level of Fig. 2A and Fig. 2B. However, Gilchriest[3] and Schilling[8] conclude that the PLL with BPL has a higher threshold level than the PLL without BPL.

There are intuitive arguments for both positions. For the latter position one can argue that since the BPL has a threshold which occurs near the point that its input S/N = 0 db., if $B_{IF} >> B_\ell$, then the threshold of the BPL will occur at a higher S/N than that of the PLL, and hence will dominate and cause the total system to have a higher threshold than the PLL alone. For the former position one can argue that the BPL removes only the in-phase component of the noise and since the PLL is a phase demodulator, the absence of this noise term should not increase its threshold. Of course the arguments are intuitive, therefore the question remains open as to whether Fig. 1 has the same threshold as Fig. 2B. With a great deal more effort it may be possible to answer this question using the Voltera expansion[12] to determine the phase error variance of the PLL with BPL.

VI. McRae's System with a Sawtooth Non-linearity

The sawtooth non-linearity will now be considered. This choice is motivated by the existence of PLL's with sawtooth characteristic phase detectors. This type of PLL or so called "lin-lock" loop has been investigated by Byrne[1] and Splitt[9].

First the sinusoidal non-linearity of Fig. 1 is replaced by the sawtooth function

$$\begin{aligned} f(\alpha') &= \alpha' \quad \forall\, \alpha' \ni -\pi < \alpha' < \pi \\ f(\alpha'+2\pi) &= f(\alpha') \quad \forall\, \alpha \text{ on real line} \end{aligned} \tag{32}$$

Then an attempt is made to manipulate the system equations into a form that makes them comparable with those of a PLL with an appropriate phase detector characteristic.

If the gain g, of the integrator is removed from the integrator block and included in the loop non-linearity block, a new equivalent function for the non-linearity can be defined by replacing α' by $g\,\alpha$. Thus

$$\begin{aligned} f(\alpha) &= g\,\alpha \quad \forall\, \alpha \ni -\pi < g\,\alpha < \pi \\ f(\alpha+2\pi/g) &= f(\alpha) \quad \forall\, \alpha \text{ on real line.} \end{aligned} \tag{33}$$

This new function has a period of $2\pi/g$.

Since integration is a linear operation

$$\int_0^t (\emptyset_1 - \dot{\theta}_o)\, dt + \emptyset_1 - \theta_1 = \int_0^t \dot{\emptyset}\, dt + \emptyset_1 + \int_0^t \theta_o\, dt + \theta_1 \tag{34}$$

If $g^{-1} = n$ is restricted to real integer values, the result is the block diagram in Fig. 3.

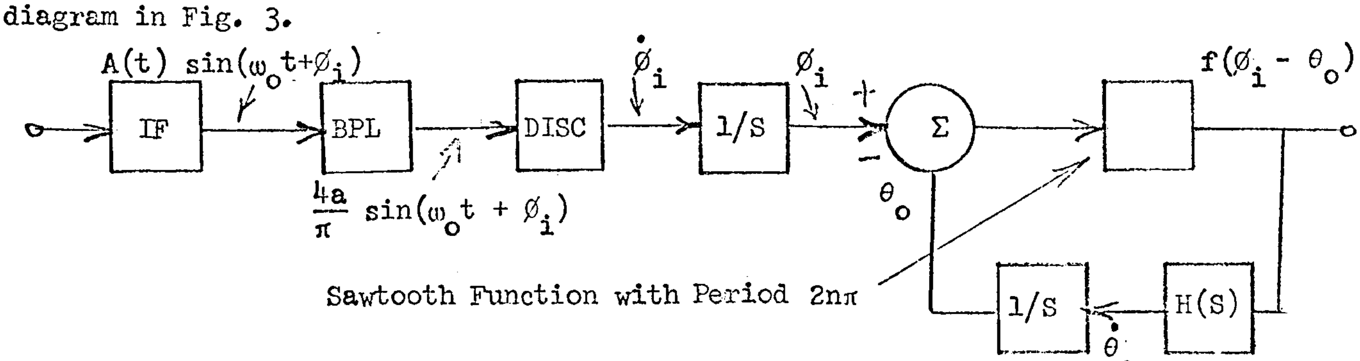

Fig. 3. Modified McRae System With Sawtooth Function

VII. Comparison of the Modified McRae System with the PLL

If the bandwidths $B_{IF} << \omega_o/n$ and $B_H << \omega_o/n$ the two integrators, the adder, and the non-linearity can be approximated asymptotically for large ω_o/B_{IF} by a pulse generating VCO, VCO_λ, called a λ function generator, a sawtooth generating VCO, VCO_S, and a sample and hold circuit as shown in Fig. 4. The mathematical justification for this follows.

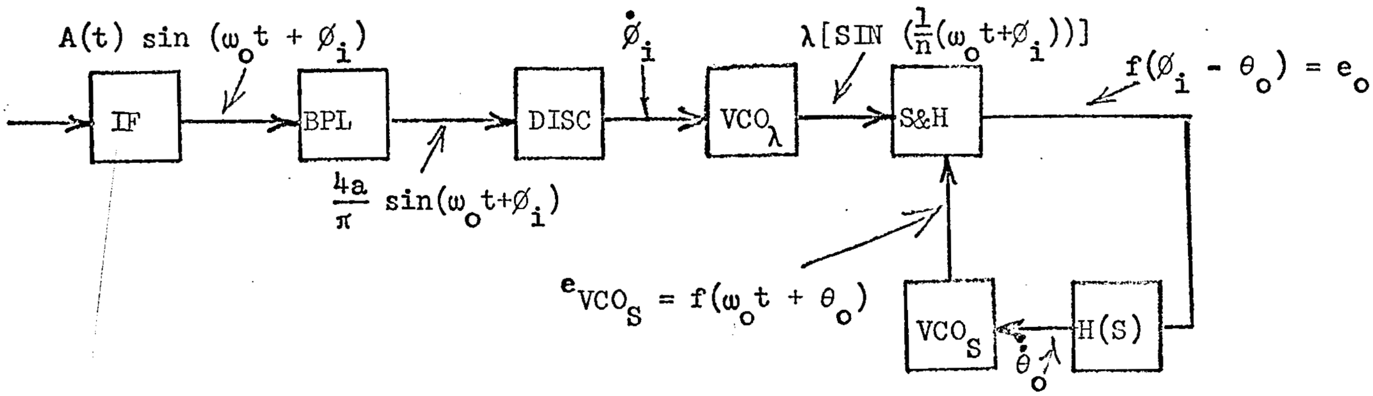

Fig. 4. Proposed Equipment System to Fig. 3

If the input to the VCO_λ is $\dot{\emptyset}(t)$, the output is defined as

$$e_{VCO} = \lambda \left[\sin \left(\frac{1}{n} (\omega_o t + \gamma (t))\right)\right] \tag{35}$$

where λ is defined as

$$\lambda(\tau) = \begin{cases} 1 & \forall\ \tau \ni \tau = 0 \text{ and } \dot{\tau} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

Therefore a λ pulse is generated each time the sine function has a positive-going zero crossing.

The sample and hold circuit, S and H, is described by its input/output relationship

$$e_o(t) = e_{VCO_S}(t_j) \quad \forall\ t \ni t_j < t < t_{j+1}$$

where j goes over all real integers and t_j are the times for which $\lambda = 1$.

The VCO_S is described by its input/output relationship,

$$e_{VCO_S} = f[\omega_o t + \int_o^t \dot{\theta}_o\ (\lambda)\ d\lambda + \theta_1]\ \forall\ t > 0 \tag{37}$$

This function can be realized physically quite easily by using a unijunction relaxation oscillator in which the capacitor charging current is controlled so that it is proportional to $\omega_o + \dot{\theta}_o$ and for which the center frequency is ω_o/n when the input is $\dot{\theta}_o(t)$.

Under these conditions the output of the sample and hold circuit is

$$f[g\ \dot{\theta}_o\ (t_j)]\ \lambda\ t \ni t_j < t < t_j + 1 \text{ and } \forall j \text{ real integer.} \tag{38}$$

Since the t_j are the times when $\sin\ [\frac{1}{n}(\omega_o t + \emptyset_i)]$ has positive zero crossing,

$[\omega_o\ t_j + \emptyset_i(t_j)]$ (Modulo $2n\pi$) = 0 $\forall j$ real integer.

If $B_H << \omega_o/n$ then $|\ [\theta_o(t_j) - \theta_o(t_{j+m})| << \omega_o(t_j - t_{j+m})/n$ and for any m

$$\lim_{nB_H/\omega_o \to 0} \left[\frac{\theta_o(t_j) - \theta_o(t_{j+m})}{\omega_o(t_j - t_{j+m})}\right] = 0 \tag{39}$$

because the correlation time of the loop filter is long compared to n/ω_o. Similarly if $B_{IF} << \omega_o/n$ then $|\ [\emptyset_i(t_j) - \emptyset_i(t_{j+m})| << \omega_o(t_j - t_{j+m})/n$ and for any m

$$\lim_{nB_{IF}/\omega_o \to 0} \left[\frac{\emptyset_i(t_j - \emptyset_i(t_{j+m})}{\omega_o(t_j - t_{j+m})}\right] \tag{40}$$

because the correlation time of the IF amplifier is long compared to n/ω_o. Therefore the sampling of e_{VCO} will occur at times that are asymptotically periodic for large $\omega_o/n\ B_{IF}$ and e_{VCO} will approach a sawtooth wave with constant period asymptotically for large $\omega_o/n\ B_\ell$.

Under these conditions the output of the S and H is asymptotically constant and equal to

$$f[\phi(t) - \theta_o(t)] \tag{41}$$

since

$$\phi_i(t_j) - \theta_o(t_j) = \phi_i(t) - \theta_o(t) = \phi_i(t_{j+m}) - \theta_o(t_{j+m}) \tag{42}$$

$$\forall\ t \ni t_j < t < t_{j+m}$$

in the limit for large $\omega_o/n\ B_{IF}$ and $\omega_o/n\ B_\ell$. Note that (42) does not hold if a cycle slip occurs. However, the cycle slip still appears in (41) but it will be delayed because of the sample time by an amount small compared to the correlation time. Thus the validity of (41) is preserved for all S/N. Since the output of the non-linearity of Fig. 3 is the same function as the output of S and H of Fig. 4, these two diagrams describe systems that are equivalent for all S/N.

Now the discriminator and VCO_λ function are combined. Since the output of the VCO_λ is

$$e_{VCO} = \lambda\ [\sin (\tfrac{1}{n}\ (\omega_o t + \phi_i))] \tag{43}$$

for a discriminator input of $\frac{4a}{\pi} \sin (\omega_o t + \phi_i)$, they may be replaced by a frequency divider with divisor, n, followed by a λ function generator. This can be physically implemented by using a digital counter which counts the input positive zero crossings up to a total of n, then resets itself to zero and generates a sampling pulse that is narrow compared to the loop filter correlation time. Then this cycle repeats itself again continuously. Byrne[1] used a divider along with a flip-flop phase detector to create a PLL with a sawtooth phase detector characteristic with period $2n\pi$.

Since these transformations are valid for all S/N, the result, a BPL followed by a PLL with modulo $2n\pi$ linear phase detector characteristic, is equivalent to McRae's system as modified to have the sawtooth non-linearity. Therefore they have the same threshold and noise performance.

The author has experimentally shown[6] that a sample and hold circuit, when used with a sawtooth generating VCO in a PLL, for the case n = 1, does have a characteristic that can be approximated by $f(\phi_i - \theta_o)$ under the above bandwidth conditions when -10 db $<$ S/N $<$ 40 db. (Referred to the loop bandwidth).

The comparison of the modified McRae system with the "lin-lock" loop without preceeding BPL does not appear to be tractable for the same reason as was mentioned on pp. 9-10.

VIII. Tan-Lock Non-Linearity Where g = 1

The "tan-lock" non-linearity will now be considered. The "tan-lock" non-linearity was chosen because of the work done by Robinson[7] and Uhran[11] where the use of this non-linearity as a phase detector characteristic was investigated.

It is felt that it is only useful to treat the case of $g = 1$ since this is the only case that is known to have been experimentally implemented.

The "tan-lock" non-linearity for constant S + N envelope, α such as the BPL output is

$$f(\phi_i - \theta_o) = \frac{\alpha(1 + k)\ \sin(\phi_i - \theta_o)}{1 + k\ \alpha\ \cos\ (\phi_i - \theta_o)} \tag{44}$$

and k is the "tan-lock" coefficient which is such that $0 < k < 1$. To prevent delta functions due to the denominator going thru zero, it is useful to have α, the input envelope, equal to 1. Fig. 5 is a modification of the McRae system with "tan-lock" characteristic f.

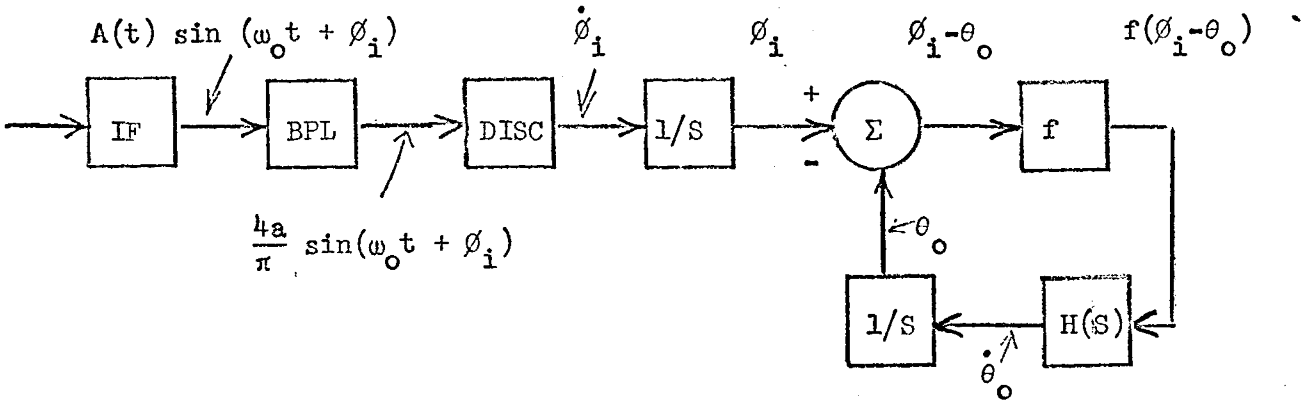

Fig. 5. Modified McRae System With "Tan-lock" Non-linearity f.

The equation of the non-linear loop filter in Fig. 5 is

$$\dot{\theta}_o = h(t) * f\ (\phi_i - \theta_o) \tag{45}$$

Consider the system in Fig. 6.

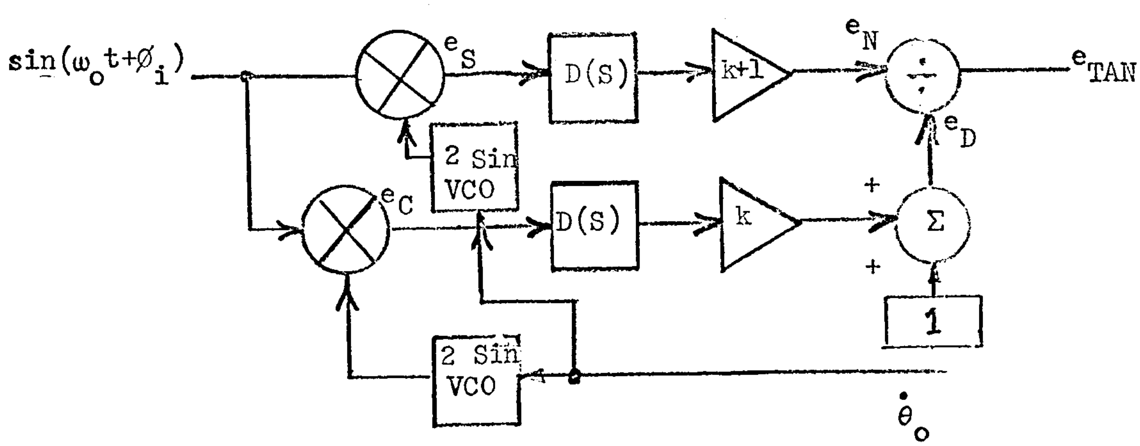

Fig. 6. Tan-lock Characteristic Generator

Here each D(S) is a low pass filter which has a bandwidth B_D such that

$$B_\ell << B_D << 2\omega_o \tag{46}$$

The output of the upper multiplier is

$$\begin{aligned} e_S &= \sin\ (\omega_o t + \phi_i) \cdot 2 \cdot \cos\ (\omega_o t + \theta_o) \\ &= \sin\ (\phi_i - \theta_o) + \sin\ (2\omega_o t + \phi_i + \theta_o) \end{aligned} \tag{47}$$

and the output of the lower multiplier is

$$e_c = 2 \sin (\omega_o t + \phi_i) \sin (\omega_o t + \theta_o)$$

$$= \cos (\phi_i - \theta_o) - \cos (\phi_i + \theta_o + 2 \omega_o t).$$

It is assumed that the frequency roll off characteristic of D(S) is sufficiently fast as well as equation (46) being satisfied so that their response at the frequency $2\omega_o$ can be neglected. It is also assumed that when used in a PLL with loop bandwidth B_ℓ the phase shift and roll off of these filters can be neglected for frequencies up to B_ℓ if the outputs of D(S) are $\sin(\phi_i - \theta_o)$ and $\cos(\phi_i - \theta_o)$ respectively. Then the output, e_{TAN}, is $f(\phi_i - \theta_o)$, the "tan-lock" function. However, Fig. 6 is also recognized as the implementation of the "tan-lock" characteristic for a PLL. Since the inputs to the "tan-lock" characteristic generator are $\frac{4a}{\pi} \sin (\omega_o t + \phi_i)$, and $\dot{\theta}_o$ and the output is $f(\phi_i - \theta_o)$, it is mathematically valid to replace the discriminator, the two integrators, the adder and non-linear function, f, by the contents of Fig. 6 since it also has the inputs $\frac{4a}{\pi} \sin (\omega_o t + \phi_i)$ for the case $a = \pi/4$, and $\dot{\theta}_o$ and output $f(\phi_i - \theta_o)$.

Since the assumptions made in Fig. 5 and Fig. 6 are valid for all S/N, the resulting system of BPL followed by PLL with "tan-lock" phase detector is equivalent to McRae's system with "tan-lock" non-linearity post discriminator filter for the case $g = 1$. Therefore both systems have the same noise performance and the same threshold behaviour.

The comparison of the modified McRae system with the "tan-lock" loop without preceding BPL does not appear to be tractable for the same reason as was mentioned on pp. 9-10.

IX. Conclusions

In summary it has been shown that 1) McRae's system with appropriate non-linearity is mathematically equivalent to the PLL when preceded by a BPL for the following non-linearities:

a. Modulo 2π sine function phase detector characteristic.

b. Modulo $2n\pi$ "lin-lock" phase detector characteristic.

c. Modulo 2π "tan-lock" phase detector characteristic.

and 2) that the noise threshold of McRae's system is the same as that of a PLL preceded by BPL for the above set of non-linearities.

Still to be determined are the threshold of the "lin-lock" loop and the effect of the BPL upon a PLL.

REFERENCES

(1) Byrne, C. J., "Properties and Design of the Phase Controlled Oscillator with a Sawtooth Comparator," Bell System Technical Journal, March, 1962, pp. 559-663.

(2) Develet, Jean A., Jr., "An Analytic Approximation of Phase-Lock Receiver Threshold," IEEE Transactions on Space Electronics and Telemetry, March, 1963, p.9.

(3) Gilchriest, C. E., "Application of Phase-Locked Loop to Telemetry as a Discriminator or Tracking Filter," IRE Transactions on Telemetry and Remote Control, June, 1958, p.20.

(4) Margolis, Stephen G., "The Response of a Phase-Locked Loop to a Sinusoid Plus Noise," IRE Transactions on Information Theory, June, 1957, p.136.

(5) McRae, Daniel, Guy Pelchat, and Earl Smith, "FM Threshold Lowering by Non-Linear Video Filtering," Internal Report, 1965, Radiation, Inc.

(6) Olsen, Donald P., "A 4th Order Phase-Lock Tracking Filter," Brigham Young University, Masters Thesis, August, 1964.

(7) Robinson, L. M., "A Phase Lock Loop of Extended Tracking Capability," 1966 National Winter Conference on Military Electronics, p.p.396-421.

(8) Schilling, D. L., "On the Threshold Extension Capability of the PLL and FDMFB," Proceedings IEEE, May, 1964,

(9) Splitt, Frank G., "Design and Analysis of a Phase Locked Loop of Wide Dynamic Range," IEEE Transactions of Communication Technology, August, 1966, p.432.

(10) Tausworthe, R. C., "A Method for Calculating Phase-Locked Loop Performance Near Threshold," 1966 National Telemetering Conference Proceedings, BP-2.3, p.249.

(11) Uhran, J. J., "Threshold Study of Phase Lock Loop Systems," Purdue University Electronic Systems Laboratory, TR-EE-66-19, December, 1966.

(12) Van Trees, H. L., "Functional Techniques for the Analysis of the Non-Linear Behaviour of Phase-Locked Loops," Proceedings IEEE, August, 1964, pp.894-911.

(13) Viterbi, A. J., "Phase Lock Loop Dynamics in the Presence of Noise by Fokker Planck Techniques," Proceedings IEEE, December, 1963, pp.1737-1753.

(14) Davenport and Root, "Random Signals and Noise," McGraw Hill; 1958, pp. 280-288.